CSE 392: Matrix and Tensor Algorithms for Data	Spring 2024
Lecture 24 — 2024.04.15	
Instructor: Shashanka Ubaru	*Scribe: Xuxi Chen*

1 Introduction

1.1 What are Tensor Networks?

- A network that works on tensors
- Alternative formulation to the standard, cumbersome algebraic tensor representation
- Conceived by Roger Penrose in 1971 "It now ceases to be important to maintain a distinction between upper and lower indices"
- Instrumental in tensor computation and analysis
- Nodes (or vertices) represent individual tensors
- Edges are (typically) non-directed and represent tensor index
- Connected (standard) edges represent (Einstein) summation over an index
- Free (dangling) indices depicted as edges attached to a single vertex
- Self-connecting edge (from a tensor to itself) represents trace operation
- Number of edges on nodes indicate the order of the tensor

1.2 Tensor Networks Applications

The examples are shown in Figure 1.

1.3 How Powerful are Tensor Networks?

- Tensor networks invariants / isomorphism offers means to analyze and identify (space and time complexity) structure in high dimensional computation
- Such embarrassment can happen to anyone, unless one appreciates the power of tensor networks...

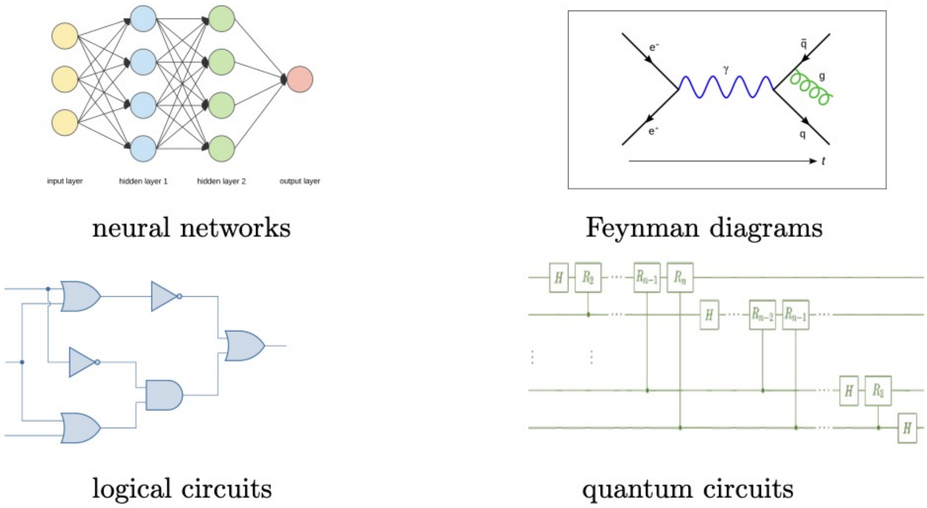
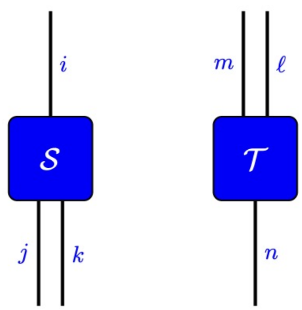


Figure 1: Some examples of tensor networks

2 Tensor Network

2.1 Tensor Network - Tensor Product

- Multiple disconnected tensors in the same diagram → multiplied by tensor product



$$\mathcal{S}^{i}_{jk} \otimes \mathcal{T}^{k\ell}_{m}$$

Figure 2: Demonstration of Tensor Product

2.2 Tensor Networks Invariants - Planar Deformation

- What is the difference between the networks in Figure 3?
- These networks are **isomorphic**.
- Tensors can freely roam past each other (planar deformation)

$$(\mathbb{1} \otimes B)(A \otimes \mathbb{1}) = A \otimes B = (A \otimes \mathbb{1})(\mathbb{1} \otimes B)$$

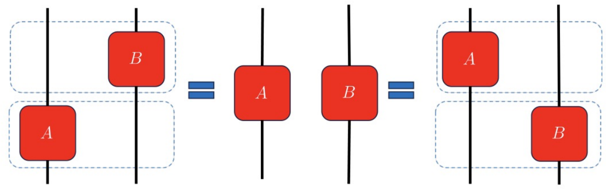


Figure 3: Isomorphic networks.

2.3 Tensor Networks Invariants - Planar Deformation

The networks are equivalent in Figure 4.

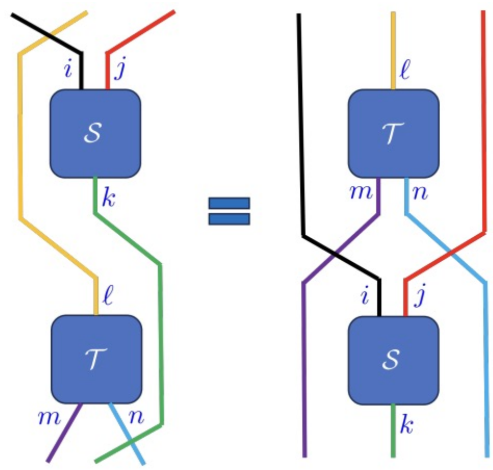


Figure 4: Equivalent networks.

2.4 Tensor Networks Transposition

Edge swapping is akin to index swap (transposition in matrices) as demonstrated in Figure 5.

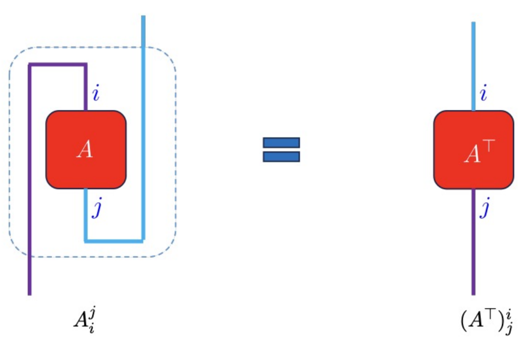


Figure 5: Tensor Networks Transposition - Index Swap

Tensor networks are indifferent to edges "detours" as demonstrated in Figure 6.

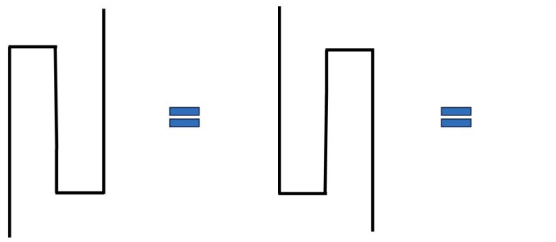


Figure 6: Tensor Networks Transposition - Detours

2.5 Tensor Networks - Penrose Duality

- **Penrose Duality** - bijection induced by bending wires
- Specific tensors (wire, cup, cap) play the role of **Kronecker's delta** and enable:
  - Tensor index contraction by diagrammatic connection
  - Raising and lowering indices
  - Represent duality between maps, states and linear transformations

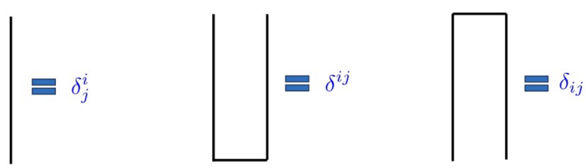


Figure 7: Penrose Duality

2.6 Anti-Symmetry

- A tensor is fully anti-symmetric if swapping any pair of indices changes its sign.
- For example in 2D:

$$A_{ij} = -A_{ij}$$

- The ϵ_{ij} tensor is used to represent the fully anti-symmetric Levi-Civita symbol

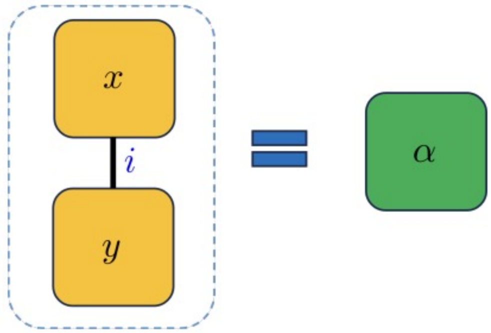
$$\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

3 Tensor Networks Contractions

3.1 Tensor Networks Contractions - Vector-Vector

- How do tensors interact?

- Represents a dot-product between two vectors which entails a scalar.
- Edge contraction implies summation over the joint index.

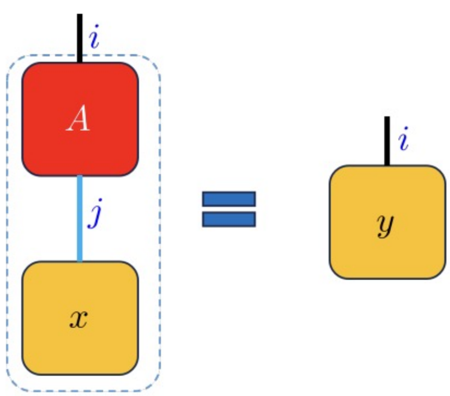


$$\sum_i x_i y^i = \alpha$$

Figure 8: Tensor Networks Contractions - Vector-Vector

3.2 Tensor Networks Contractions - Matrix-Vector

- How does a matrix and a vector contract?
- Matrix-vector from a tensor network perspective is effectively a vector.

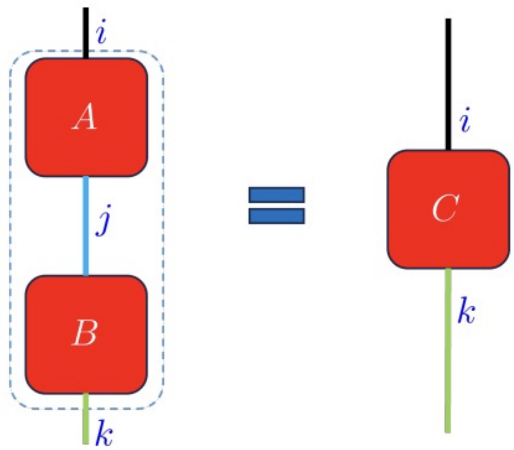


$$\sum_j A^i_j x^j = y^i$$

Figure 9: Tensor Networks Contractions - Matrix-Vector

3.3 Tensor Networks Contractions - Matrix-Matrix

- Similarly, matrix-matrix contraction over a single edge entails a matrix (matrix-matrix product).

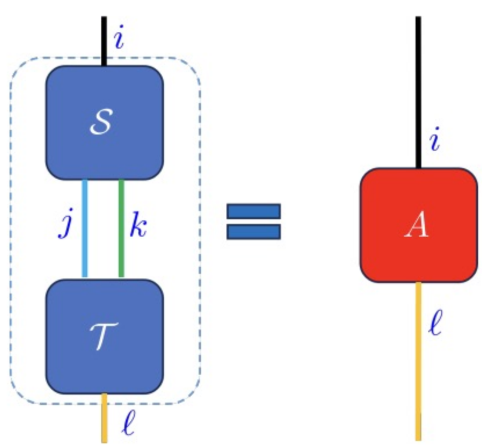


$$\sum_j A^i_j B^j_k = C^i_k$$

Figure 10: Tensor Networks Contractions - Matrix-Matrix

3.4 Tensor Networks Contractions - Tensor-Tensor

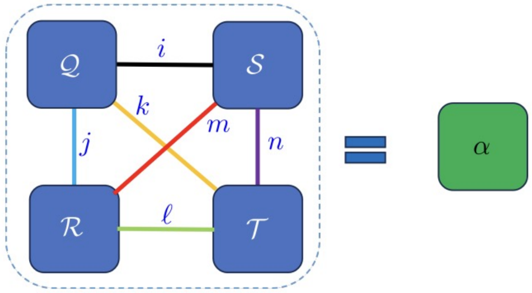
- How do tensors interact with other tensors?
- Two 3^{rd} degree tensors contracted by 2 indices form a matrix



$$\sum_{jk} \mathcal{S}^i_{jk} \mathcal{T}^{jk}_\ell = A^i_\ell$$

Figure 11: Tensor Networks Contractions - Tensor-Tensor

- What would such contraction yield?
- Four 3^{rd} order tensors, where all edges contracted, entails a scalar.

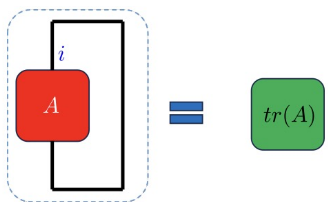


$$\sum_{ijk\ell mn} \mathcal{Q}^{i}_{jk}\mathcal{R}^{j}_{\ell m}\mathcal{S}^{mn}_{i}\mathcal{T}^{k\ell}_{n} = \alpha$$

Figure 12: Tensor Networks Contractions - Tensor-Tensor

3.5 Tensor Networks Contractions - Trace

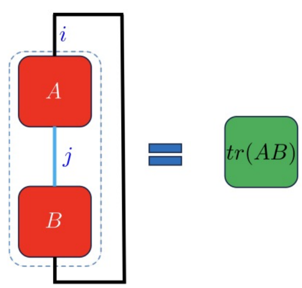
- What contraction of a tensor to itself means?



$$\sum_{i} A^{i}_{i} = tr(A)$$

Figure 13: Tensor Networks Contractions - Trace

- What contraction of a matrix product to itself means?

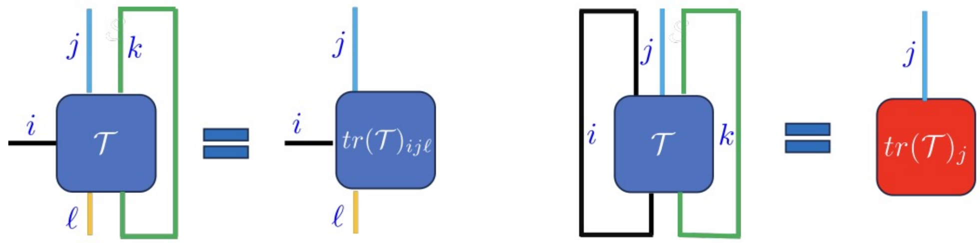


$$\sum_{ij} A^{i}_{j}B^{j}_{i} = tr(AB)$$

Figure 14: Tensor Networks Contractions - Trace

3.6 Tensor Networks Contractions - Partial Trace

- What partial contraction of a tensor product to itself means?



$$\sum_{k} \mathcal{T}_{\ell}^{ijk} = tr(\mathcal{T})_{\ell}^{ij} \qquad \sum_{ik} \mathcal{T}^{ijk} = tr(\mathcal{T})^{j}$$

Figure 15: Tensor Networks Contractions - Partial Trace

Tensor Networks Contractions - Trace Cyclicity

- How can we prove trace cyclicity?
- Trivially proven with tensor networks due to rotational invariance of the network (graph isomorphism).

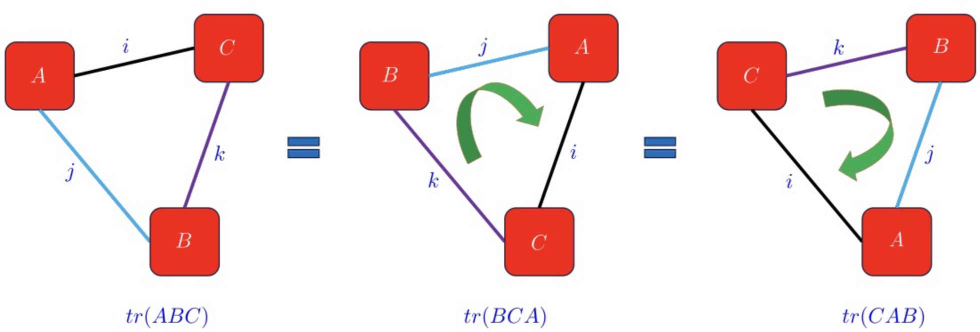


Figure 16: Tensor Networks Contractions - Trace Cyclicity

- How is it related to sketching and low rank approximations and tensor algebra?
- Split - inverse form of tensor contraction

3.7 Tensor Networks - Splits and Low-Rank Approximation

- Can we always split?
  - Can always compute SVD on matrices.
- How do we extend this to tensors?
  - Vectorize and then employ matrix SVD

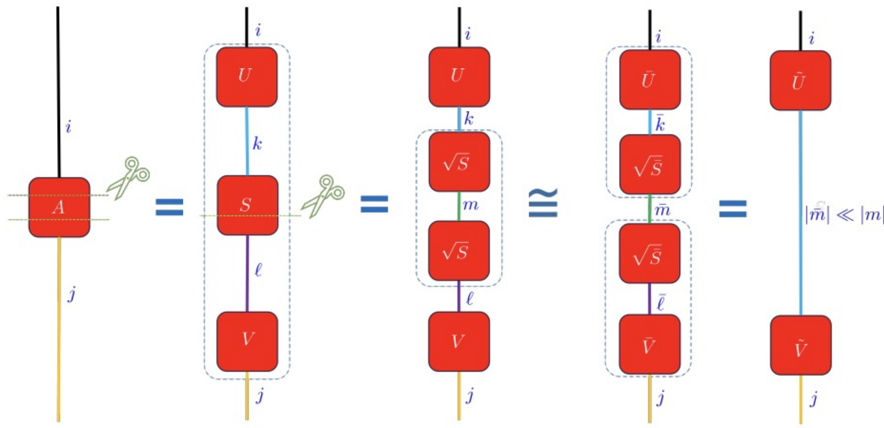


Figure 17: Split

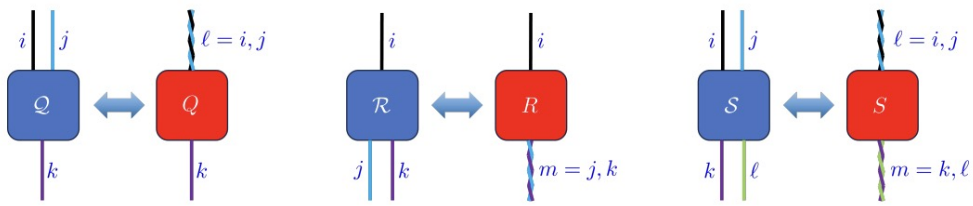


Figure 18: Vectorize and then employ matrix SVD

– And conduct native tensor decomposition ...

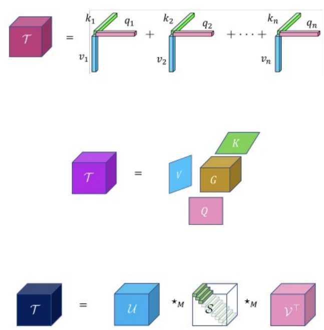


Figure 19: Tensor Decompositions