CSE 392/CS 395T/M 397C: Matrix and Tensor Algorithms for Data

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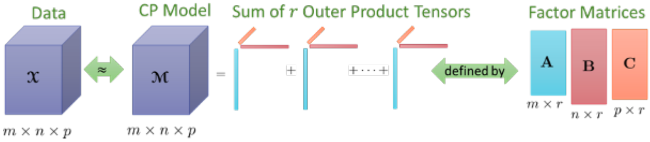
Lecture 17: Randomized CP - I

Outline

1. CP-ALS
2. CP-ARLS
3. CP-ARLS-Mix
    - Kronecker FJLT

Alternating Least Squares (CP-ALS)



$$\min_{\mathbf{A},\mathbf{B},\mathbf{C}} \|\mathcal{X} - [\![\mathbf{A},\mathbf{B},\mathbf{C}]\!]\|_F$$

General Idea: solve for ONE matrix, holding the others fixed.

- **CP-ALS:** Repeat until converged…
  - Solve for $\boldsymbol{A}$ (with $\boldsymbol{B}$ and $\boldsymbol{C}$ fixed)
  - Solve for $\boldsymbol{B}$ (with $\boldsymbol{A}$ and $\boldsymbol{C}$ fixed)
  - Solve for $\boldsymbol{C}$ (with $\boldsymbol{A}$ and $\boldsymbol{B}$ fixed)

Special Structure of Least Squares Problem

$$\min_{\boldsymbol{A}} \|\boldsymbol{X}_{(1)} - \boldsymbol{A}(\boldsymbol{C} \odot \boldsymbol{B})^\top\|_F^2$$

$$\min_{\boldsymbol{A}} \|(\boldsymbol{C} \odot \boldsymbol{B})\boldsymbol{A}^\top - \boldsymbol{X}_{(1)}^\top\|_F^2$$

By normal equations:

$$(\boldsymbol{C} \odot \boldsymbol{B})^\top(\boldsymbol{C} \odot \boldsymbol{B})\boldsymbol{A}^\top = (\boldsymbol{C} \odot \boldsymbol{B})^\top \boldsymbol{X}_{(1)}^\top$$

$$(\boldsymbol{C}^\top\boldsymbol{C} * \boldsymbol{B}^\top\boldsymbol{B})\boldsymbol{A}^\top = (\boldsymbol{C} \odot \boldsymbol{B})^\top \boldsymbol{X}_{(1)}^\top$$

$$\boldsymbol{A}^\top = (\boldsymbol{C}^\top\boldsymbol{C} * \boldsymbol{B}^\top\boldsymbol{B})^{-1}(\boldsymbol{C} \odot \boldsymbol{B})^\top \boldsymbol{X}_{(1)}^\top$$

$$\boldsymbol{A} = \boldsymbol{X}_{(1)}(\boldsymbol{C} \odot \boldsymbol{B})(\boldsymbol{C}^\top\boldsymbol{C} * \boldsymbol{B}^\top\boldsymbol{B})^{-1}$$

Special Structure of Least Squares Problem (d-way)

$$\min_{\boldsymbol{A}_k} \|\boldsymbol{X}_{(k)} - \boldsymbol{A}_k(\underbrace{\boldsymbol{A}_d \odot \cdots \odot \boldsymbol{A}_{k+1} \odot \boldsymbol{A}_{k-1} \odot \cdots \odot \boldsymbol{A}_1}_{\boldsymbol{Z}_k})^\top\|_F^2$$

$$\min_{\boldsymbol{A}_k} \|\boldsymbol{Z}_k\boldsymbol{A}_k^\top - \boldsymbol{X}_{(k)}^\top|_F^2$$

$$\boldsymbol{Z}_k^\top \boldsymbol{Z}_k \boldsymbol{A}_k^\top = \boldsymbol{Z}_k^\top \boldsymbol{X}_{(k)}^\top$$

$$(\underbrace{\boldsymbol{A}_d^\top \boldsymbol{A}_d * \cdots * \boldsymbol{A}_{k+1}^\top \boldsymbol{A}_{k+1} * \boldsymbol{A}_{k-1}^\top \boldsymbol{A}_{k-1} \cdots * \boldsymbol{A}_1^\top \boldsymbol{A}_1}_{\boldsymbol{V}_k})\boldsymbol{A}_k^\top = \boldsymbol{Z}_k^\top \boldsymbol{X}_{(k)}^\top$$

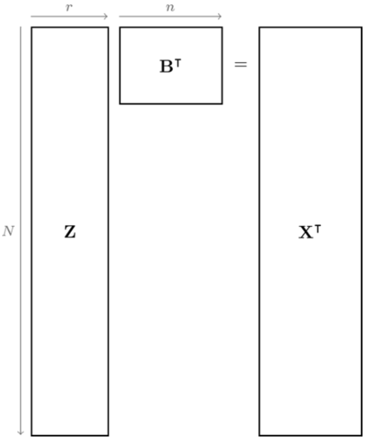
$$\boldsymbol{A}_k = \boldsymbol{X}_{(k)} \boldsymbol{Z}_k \boldsymbol{V}_k^{-1}$$

CP-ALS Full Algorithm

Inputs: Tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$, desired rank $r \in \mathbb{N}$.

```
Initialize $\boldsymbol{A}_k \in \mathbb{R}^{n_k \times r}$ for all $k \in [d]$
repeat
    for $k = 1, \ldots, d$ do
        $\boldsymbol{Z}_k \leftarrow \boldsymbol{A}_d \odot \cdots \odot \boldsymbol{A}_{k+1} \odot \boldsymbol{A}_{k-1} \odot \cdots \odot \boldsymbol{A}_1$
        $\boldsymbol{A}_k \leftarrow \arg\min_{\boldsymbol{B}} \|\boldsymbol{Z}_k \boldsymbol{B}^\top - \boldsymbol{X}_{(k)}^\top\|_F^2$
    end
until $\|\mathcal{X} - [\![\mathbf{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_d]\!]\|_F^2$ ceases to decrease
```

Can randomization help?



$$\min_{\boldsymbol{B}} \|\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{X}^\top\|_F^2$$

Conversion from tensor problem. . .

$$N = \prod_{\ell=1,\neq k}^{d} n_\ell, \quad n = n_k$$

$$\boldsymbol{Z} = \boldsymbol{A}_d \odot \cdots \odot \boldsymbol{A}_{k+1} \odot \boldsymbol{A}_{k-1} \odot \cdots \odot \boldsymbol{A}_1$$

$$\boldsymbol{X} = \boldsymbol{X}_{(k)}$$

$$\boldsymbol{B} = \boldsymbol{A}_k$$

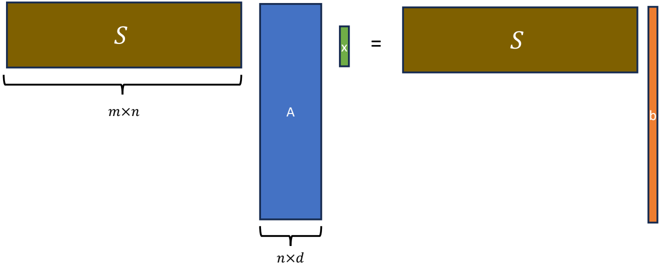
Recall: Sketch and solve

Use *Sketching*:

- Generate a sketching matrix $\boldsymbol{S} \in \mathbb{R}^{m \times n}$.
- Compute sketches $\boldsymbol{S}\boldsymbol{A}$ and $\boldsymbol{S}\boldsymbol{b}$.
- Solve:

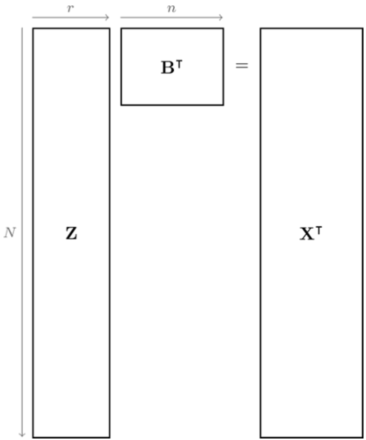
$$\tilde{\boldsymbol{x}} = \min_{\boldsymbol{x} \in \mathbb{R}^d} \|\boldsymbol{S}\boldsymbol{A}\boldsymbol{x} - \boldsymbol{S}\boldsymbol{b}\|_2^2.$$

- Typically, $m = \text{poly}(d/\epsilon)$.



Uniform sampling

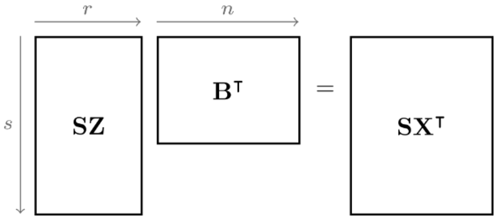
$$\min_{\boldsymbol{B}} \|\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{X}^\top\|_F^2$$



Constructing *sample* matrix $\boldsymbol{S}$ of size $s \times N$

- s be the number of samples
- Each row of $\boldsymbol{S}$ is a random row of the $N \times N$ identity matrix, Scaled by $1/\sqrt{s}$.

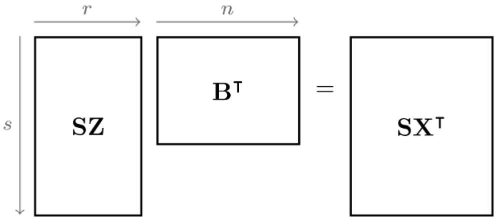
$$\min_{\boldsymbol{B}} \|\boldsymbol{S}\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{S}\boldsymbol{X}^\top\|_F^2$$



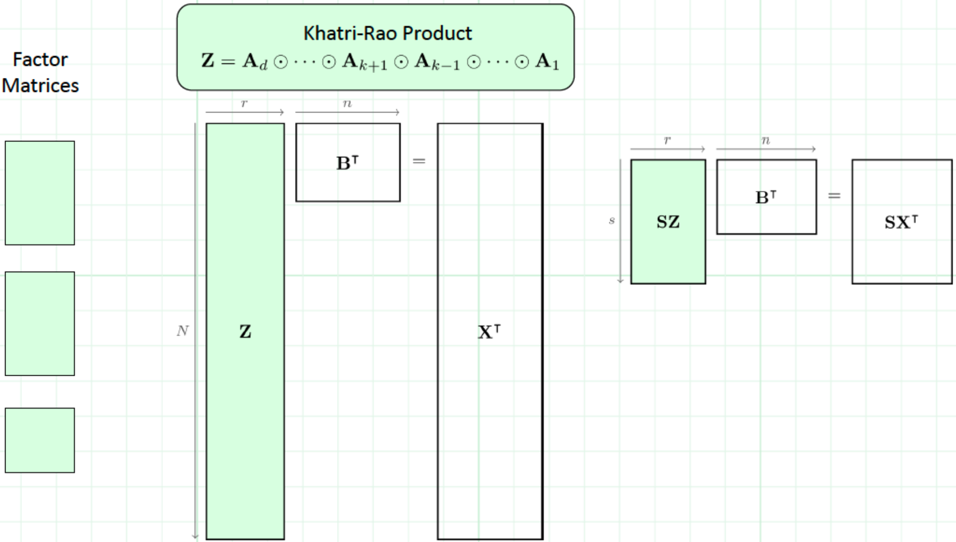
Uniform sampling

Challenges:

- Does uniform sampling “work”?
- $\boldsymbol{X}^\top$ is expensive (in memory movement) to form
- $\boldsymbol{Z}$ is expensive (in computations) to form
- Checking convergence of overall CP ALS method



Forming Sampled KRP



Factor Matrices

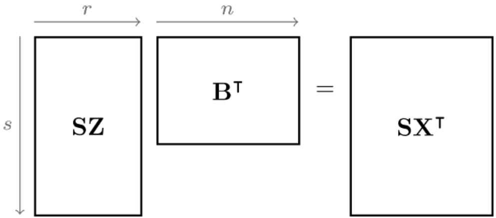
Khatri-Rao Product

$$\mathbf{Z} = \mathbf{A}_d \odot \cdots \odot \mathbf{A}_{k+1} \odot \mathbf{A}_{k-1} \odot \cdots \odot \mathbf{A}_1$$

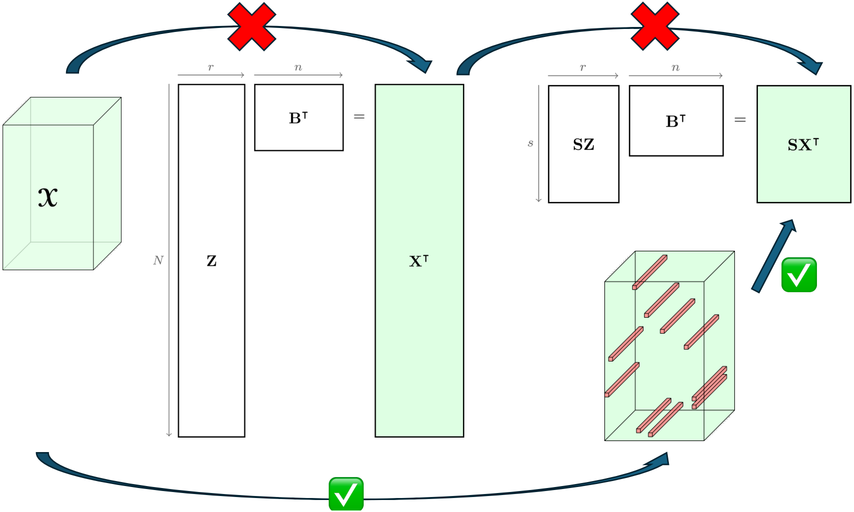
Uniform sampling

Challenges:

- Does uniform sampling “work”?
- $\boldsymbol{X}^\top$ is expensive (in memory movement) to form
- $\boldsymbol{Z}$ is expensive (in computations) to form ✓
- Checking convergence of overall CP ALS method



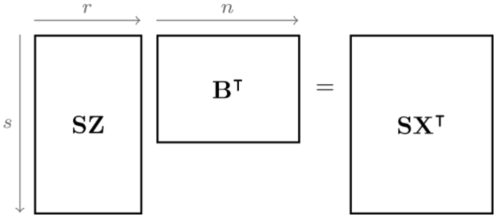
Forming Sampled Right-hand Side



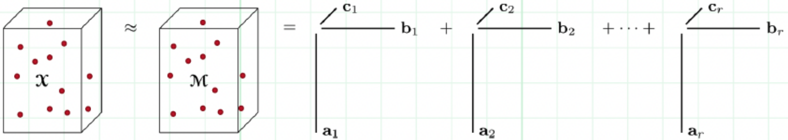
Uniform sampling

Challenges:

- Does uniform sampling “work”?
- $\boldsymbol{X}^\top$ is expensive (in memory movement) to form ✓
- $\boldsymbol{Z}$ is expensive (in computations) to form ✓
- Checking convergence of overall CP ALS method



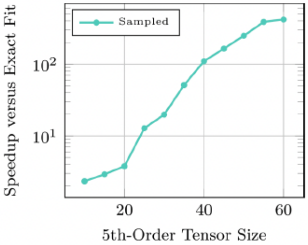
Checking Convergence



$$\min_{\mathbf{A},\mathbf{B},\mathbf{C}} \|\mathcal{X}-\mathcal{M}\|^2 \equiv \sum_{i,j,k} (x_{ijk}-m_{ijk})^2$$

$$\min_{\mathbf{A},\mathbf{B},\mathbf{C}} \|\mathcal{X}-\mathcal{M}\|^2 \approx \frac{\prod n_k}{|\Omega|} \sum_{(i,j,k)\in\Omega} (x_{ijk}-m_{ijk})^2$$

Maybe don't even check every iteration...



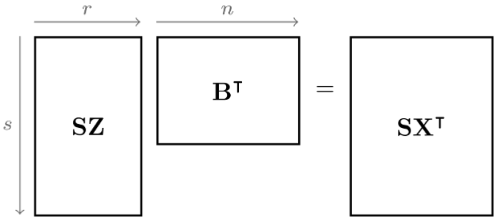
$$\frac{|f-\hat{f}|}{|f|} < 10^{-3}$$

16000 samples < 1% of full data

Uniform sampling

Challenges:

- Does uniform sampling “work”?
- $\boldsymbol{X}^\top$ is expensive (in memory movement) to form ✓
- $\boldsymbol{Z}$ is expensive (in computations) to form ✓
- Checking convergence of overall CP ALS method ✓



CP-ARLS Algorithm

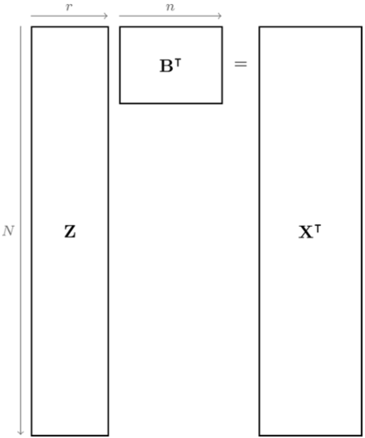
Inputs: Tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$, desired rank $r \in \mathbb{N}$, number of samples $s \in \mathbb{N}$.

```
Initialize $\boldsymbol{A}_k \in \mathbb{R}^{n_k \times r}$ for all $k \in [d]$
$\Omega \leftarrow$sampled indices for function value estimation
repeat
    for $k = 1, \ldots, d$ do
        $\boldsymbol{S} \leftarrow$ random rows of $\boldsymbol{I}$ scaled by $1/\sqrt{s}$.
        $\hat{\boldsymbol{Z}} \leftarrow \mathrm{SKRP}(\boldsymbol{S}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_{k-1}, \boldsymbol{A}_{k+1}, \ldots, \boldsymbol{A}_d)$
        $\hat{\boldsymbol{X}} \leftarrow \mathrm{STU}(\boldsymbol{S}, \mathcal{X}, k)$
        $\boldsymbol{A}_k \leftarrow \arg\min_{\boldsymbol{B}} \|\hat{\boldsymbol{Z}} \boldsymbol{B}^\top - \hat{\boldsymbol{X}}^\top\|_F^2$
    end
until SFV$(\Omega, \mathcal{X}, \mathbf{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_d)$ ceases to decrease
```

Matlab Demo

Sketching Problem with Plain Sampling

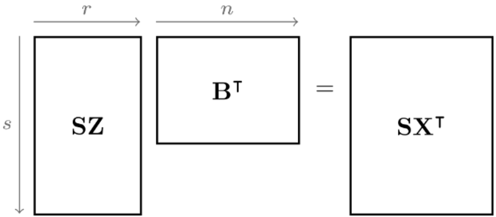
$$\min_{\boldsymbol{B}} \|\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{X}^\top\|_F^2$$



Constructing *sample* matrix $\boldsymbol{S}$ of size $s \times N$

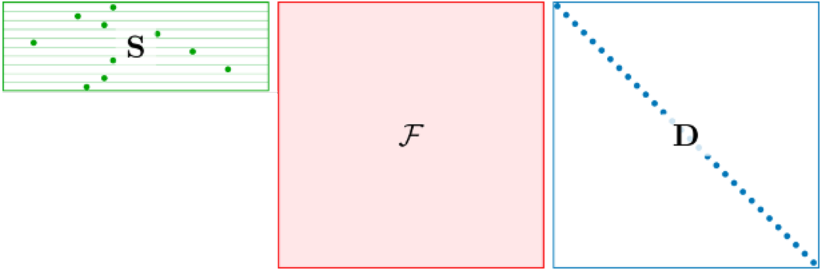
- s be the number of samples
- Each row of $\boldsymbol{S}$ is a random row of the $N \times N$ identity matrix, Scaled by $1/\sqrt{s}$.

$$\min_{\boldsymbol{B}} \|\boldsymbol{S}\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{S}\boldsymbol{X}^\top\|_F^2$$



Uniform sampling is only efficient if $\boldsymbol{Z}$ is incoherent.

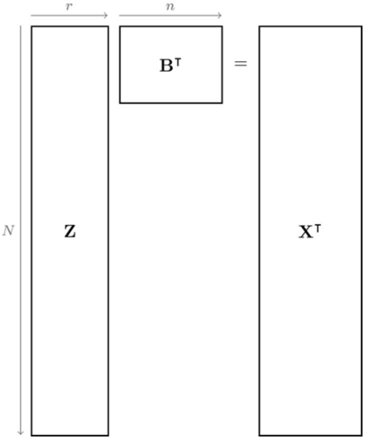
Recall: FJLT (SRHT/SRFT)



- $\boldsymbol{S}$ is a sampling matrix
- $\mathcal{F}$ is an FFT (or Hadamard) matrix.
- $\boldsymbol{D}$ is a diagonal matrix with ±1 (Radamacher) entries.

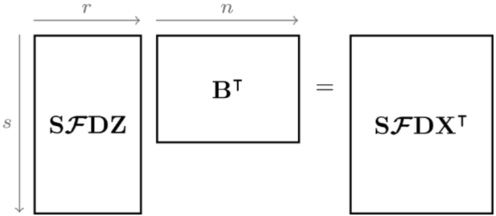
Mixing using FJLTs

$$\min_{\boldsymbol{B}} \|\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{X}^\top\|_F^2$$



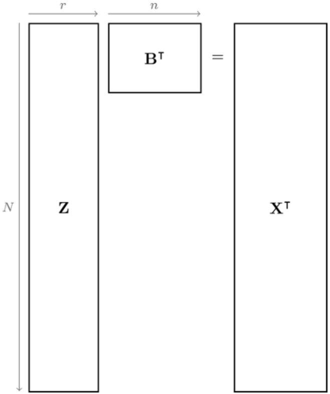
$$\min_{\boldsymbol{B}} \|\boldsymbol{S}\boldsymbol{\mathcal{F}}\boldsymbol{D}\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{S}\boldsymbol{\mathcal{F}}\boldsymbol{D}\boldsymbol{X}^\top\|_F^2$$

- $\boldsymbol{S}$ is $s \times N$ sampling matrix
- $\mathcal{F}$ is $N \times N$ FFT (or Hadamard) matrix.
- $\boldsymbol{D}$ is a $N \times N$ diagonal matrix with ± 1 (Radamacher) entries.



Mixing using Kronecker FJLTs

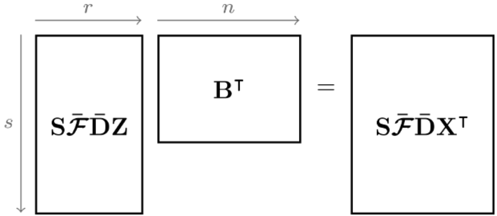
$$\min_{\boldsymbol{B}} \|\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{X}^\top\|_F^2$$



$$\boldsymbol{Z} = \boldsymbol{A}_d \odot \cdots \odot \boldsymbol{A}_{k+1} \odot \boldsymbol{A}_{k-1} \odot \cdots \odot \boldsymbol{A}_1$$

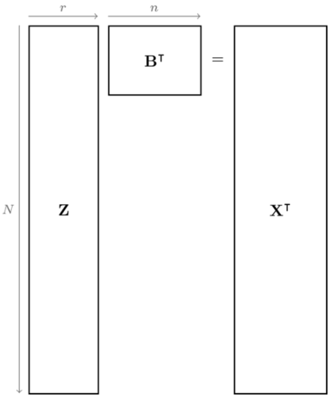
$$\min_{\boldsymbol{B}} \|\boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{X}^\top\|_F^2$$

- $\boldsymbol{S}$ is $s \times N$ sampling matrix
- $\bar{\mathcal{F}} = \mathcal{F}_d \otimes \cdots \otimes \mathcal{F}_{k+1} \otimes \mathcal{F}_{k-1} \otimes \cdots \otimes \mathcal{F}_1$.
- $\bar{\boldsymbol{D}} = \boldsymbol{D}_d \otimes \cdots \otimes \boldsymbol{D}_{k+1} \otimes \boldsymbol{D}_{k-1} \otimes \cdots \otimes \boldsymbol{D}_1$.



Kronecker FJLTs (Simpler case)

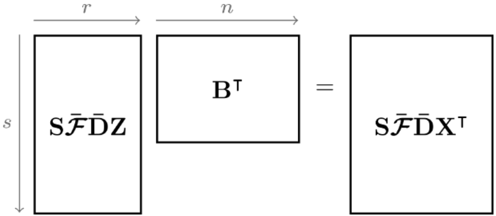
$$\min_{\boldsymbol{B}} \|\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{X}^\top\|_F^2$$



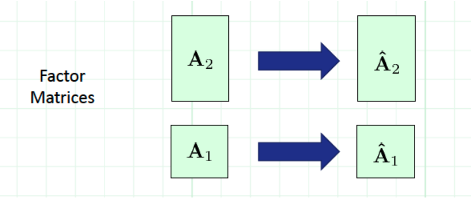
$$\boldsymbol{Z} = \boldsymbol{A}_2 \odot \boldsymbol{A}_1$$

$$\min_{\boldsymbol{B}} \|\boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{X}^\top\|_F^2$$

- $\boldsymbol{S}$ is $s \times N$ sampling matrix
- $\bar{\mathcal{F}} = \mathcal{F}_2 \otimes \mathcal{F}_1$.
- $\bar{\boldsymbol{D}} = \boldsymbol{D}_2 \otimes \boldsymbol{D}_1$.



Mixing KRP Efficiently Using Kronecker FJLT



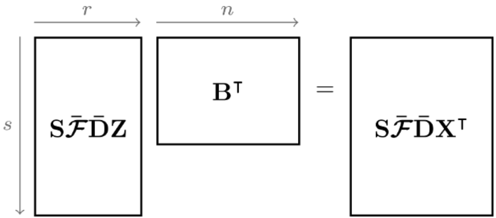
$$\begin{aligned}
\boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{Z} &= \boldsymbol{S}(\mathcal{F}_2 \otimes \mathcal{F}_1)(\boldsymbol{D}_2 \otimes \boldsymbol{D}_1)(\boldsymbol{A}_2 \odot \boldsymbol{A}_1) \\
&= \boldsymbol{S}\left((\mathcal{F}_2\boldsymbol{D}_2) \otimes (\mathcal{F}_1\boldsymbol{D}_1)\right)(\boldsymbol{A}_2 \odot \boldsymbol{A}_1) \\
&= \boldsymbol{S}\left((\mathcal{F}_2\boldsymbol{D}_2\boldsymbol{A}_2) \odot (\mathcal{F}_1\boldsymbol{D}_1\boldsymbol{A}_1)\right) \\
&= \boldsymbol{S}(\hat{\boldsymbol{A}}_2 \odot \hat{\boldsymbol{A}}_1)
\end{aligned}$$

Pre-Mixing Tensor

Need to compute sketched right hand side. . .

$$\boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{X}^\top = \boldsymbol{S}(\mathcal{F}_2 \otimes \mathcal{F}_1)(\boldsymbol{D}_2 \otimes \boldsymbol{D}_1)\boldsymbol{X}_{(3)}^\top$$

$$\min_{\boldsymbol{B}} \|\boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{Z}\boldsymbol{B}^\top - \boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{X}^\top\|_F^2$$



Pre-mixed tensor

$$\tilde{\mathcal{X}} = \mathcal{X} \times_1 \mathcal{F}_1\boldsymbol{D}_1 \times_2 \mathcal{F}_2\boldsymbol{D}_2 \times_3 \mathcal{F}_3\boldsymbol{D}_3$$

$$\tilde{\boldsymbol{X}}_{(3)}^\top = (\mathcal{F}_2\boldsymbol{D}_2 \otimes \mathcal{F}_1\boldsymbol{D}_1)\boldsymbol{X}_{(3)}^\top(\mathcal{F}_3\boldsymbol{D}_3)^\top$$

Sample before unmixing

$$\boldsymbol{S}\bar{\mathcal{F}}\bar{\boldsymbol{D}}\boldsymbol{X}^\top = (\boldsymbol{S}\tilde{\boldsymbol{X}}_{(3)}^\top)\boldsymbol{D}_3\mathcal{F}_3^*$$

CP-ARLS-Mix Algorithm

Inputs: Tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$, desired rank $r \in \mathbb{N}$, number of samples $s \in \mathbb{N}$.

```
Initialize $\boldsymbol{A}_k \in \mathbb{R}^{n_k \times r}$ for all $k \in [d]$
Draw random diagonal $\boldsymbol{D}_k$ for all $k \in [d]$
Compute $\tilde{\boldsymbol{A}}_k = \mathcal{F}_k \boldsymbol{D}_k \boldsymbol{A}_k$for all $k \in [d]$
Compute $\tilde{\mathcal{X}} = \mathcal{X} \times_1 \mathcal{F}_1 \boldsymbol{D}_1 \times_2 \mathcal{F}_2 \boldsymbol{D}_2 \cdots \times_d \mathcal{F}_d \boldsymbol{D}_d$
$\Omega \leftarrow$sampled indices for function value estimation
repeat
    for $k = 1, \ldots, d$ do
        $\boldsymbol{S} \leftarrow$ random rows of $\boldsymbol{I}$ scaled by $1/\sqrt{s}$.
        $\hat{\boldsymbol{Z}} \leftarrow \mathrm{SKRP}(\boldsymbol{S}, \tilde{\boldsymbol{A}}_1, \ldots, \tilde{\boldsymbol{A}}_{k-1}, \tilde{\boldsymbol{A}}_{k+1}, \ldots, \tilde{\boldsymbol{A}}_d)$
        $\hat{\boldsymbol{X}} \leftarrow \mathcal{F}_k^* \boldsymbol{D}_k \left( \mathrm{STU}(\boldsymbol{S}, \tilde{\mathcal{X}}, k) \right)$
        $\boldsymbol{A}_k \leftarrow \arg\min_{\boldsymbol{B}} \| \hat{\boldsymbol{Z}} \boldsymbol{B}^\top - \hat{\boldsymbol{X}}^\top \|_F^2$
        $\tilde{\boldsymbol{A}}_k \leftarrow \mathcal{F}_k \boldsymbol{D}_k \boldsymbol{A}_k$
    end
until SFV$(\Omega, \mathcal{X}, \mathbf{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_d)$ ceases to decrease
```

Randomized-CP

Further Reading:

- D. Cheng, R. Peng, I. Perros , and Y. Liu. *SPALS: Fast Alternating Least Squares via Implicit Leverage Scores Sampling* , NeurIPS'16
- C. Battaglino , G. Ballard, and T. G. Kolda. *A Practical Randomized CP Tensor Decomposition* , SIAM J. Matrix Analysis and Applications, 2018
- R. Jin, T. G. Kolda, and R. Ward. *Faster Johnson Lindenstrauss Transforms via Kronecker Products*, Information and Inference, 2020
- O. A. Malik, and S. Becker. *Guarantees for the Kronecker Fast Johnson Lindenstrauss Transform Using a Coherence and Sampling Argument*, Linear Algebra and its Applications, 2020
- M. A. Iwen , D. Needell, E. Rebrova , and A. Zare . *Lower Memory Oblivious (Tensor) Subspace Embeddings with Fewer Random Bits: Modewise Methods for Least Squares* , SIAM J. Matrix Analysis and Applications, 2021

Matlab Demo